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# A CLASS OF WELL-COVERED GRAPHS WITH GIRTH FOUR

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Abstract

A graph is well-covered if every maximal independent set is also a maximum independent set. A 1-well-covered graph G has the additional property that G-v is also well-covered for every point v in G. Thus, the 1-well-covered graphs form a subclass of the well-covered graphs. We examine triangle-free 1-well-covered graphs. Other than $C_5$ and $K_2$, a 1-well-covered graph must contain a triangle or a 4-cycle. Thus, the graphs we consider have girth 4. Two constructions are given which yield infinite families of 1-well-covered graphs with girth 4. These families contain graphs with arbitrarily large independence number.

# A CLASS OF WELL-COVERED GRAPHS WITH GIRTH FOUR

## INTRODUCTION

A set of points in a graph is *independent* if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the *independence number of G* and is denoted by *$\alpha(G)$*. A set of independent points which attains the maximum size is referred to as a *maximum independent set*. A set S of independent points in a graph is *maximal* (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [10] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a *well-covered* graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [3] proved that there are only four 3-connected cubic planar well-covered graphs. Royle and Ellingham [13] have recently completed the picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called *very well-covered* graphs. The subclass of very well-covered graphs was characterized by Staples [14] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [11] characterized bipartite well-covered graphs and Favaron [6] characterized the very well-covered graphs. Recently, Dean and Zito [4] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.


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Finbow and Hartnell [7] and Finbow, Hartnell, and Nowakowski [8] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [9].

A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph is in the class $W_2$ if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [15] showed that a well-covered graph is 1-well-covered if and only if it is in $W_2$. Since we will appeal mostly to the notion of extending two disjoint independent sets to disjoint maximum independent sets, henceforth we use the $W_2$ nomenclature instead of referring to 1-well-covered graphs.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are $C_3$, $C_4$, $C_5$, and $C_7$. We note that all complete graphs are also in $W_2$, but no complete bipartite graphs (except $K_{1,1}$) are in $W_2$. The cycles $C_3$ and $C_5$ are the only cycles in $W_2$.

## PRELIMINARY RESULTS

We assume that all graphs are connected, unless otherwise stated. The reader is referred to [1] for terminology and notation not defined here. Note that a disconnected graph is in $W_2$ if and only if each of its components is in $W_2$. Suppose G is well-covered, $G \neq K_1$. Let v be a point in G and consider the graph G-v. Since $G \neq K_1$, there exists a point $u \sim v$. Since G is well-covered, the point u is contained in a maximum independent set I in G. Clearly, v is not in I. Thus, I is also independent in G-v. Consequently, $\alpha(G-v) = \alpha(G)$ for any point v. Hence, from a result of Erdös and Gallai [5] it follows that $\alpha(G) \leq |V(G)|/2$. Thus, $W_2$ graphs inherit this bound on independence number.

Staples [15] proved that a $W_2$ graph cannot have an endpoint.

**Theorem 1.** If $G \in W_2$ and G is not complete, then $\delta \geq 2$.

If v is a point in the graph G, then denote the neighborhood of v by N(v). Let $G_v$ be the graph induced by G - { v ∪ N(v) }. Campbell [3] found the following very useful necessary condition for a graph to be well-covered.

**Theorem 2.** If a graph G is well-covered and is not complete, then $G_v$ is well-covered for all v in G. Moreover, $\alpha(G_v) = \alpha(G) - 1$.

Fortunately, we prove in Theorem 3 that we have a similar *necessary* condition for a well-covered graph to be in $W_2$. We will reference Theorem 3 several times in this paper.

**Theorem 3.** If a graph G is in $W_2$ and G is not complete, then $G_v$ is in $W_2$ for all v in G.

Proof. Let v be a point in G. Since G is not complete, then $G_v \neq \emptyset$. By Theorem 2, graph $G_v$ is well-covered and $\alpha(G_v) = \alpha(G) - 1$. Suppose $I_1$ and $I_2$ are disjoint independent sets in $G_v$. Then $I_1 \cup \{v\}$ is an independent set in G, as is $I_2 \cup \{v\}$. Since G is in $W_2$, there exists a maximum independent set $J_1 \supseteq I_1 \cup \{v\}$ such that $J_1 \cap I_2 = \emptyset$. Since $I_2 \cup \{v\}$ and $J_1$-v are disjoint independent sets in G, then there exists a maximum independent set $J_2 \supseteq I_2 \cup \{v\}$ such that $J_2 \cap (J_1\text{-}v) = \emptyset$. Hence, $J_2$-v and $J_1$-v are disjoint independent sets in $G_v$. Since $|J_i| = \alpha(G)$, then $|J_i\text{-}v| = \alpha(G) - 1$, for i = 1, 2. Thus, $J_1$-v contains $I_1$, $J_2$-v contains $I_2$, and $J_1$-v and $J_2$-v are disjoint maximum independent sets in $G_v$. So any two disjoint independent sets in $G_v$ can be extended to disjoint maximum independent sets in $G_v$. By definition of the class $W_2$, we conclude that $G_v \in W_2$. []

We prove in the following theorem that if a $W_2$ graph has a cutpoint, then the graph obtained by deleting the cutpoint is also a $W_2$ graph.

**Theorem 4.** If $G \in W_2$ and v is a cutpoint of G, then $G-v \in W_2$.

Proof. Let $H_1, H_2, \ldots, H_n$ be the components of G-v. Let $x \in V(H_1)$ and $y \in V(H_2)$ such that $x \sim v$ and $y \sim v$. By Theorem 3, the graphs $G_y = G-N[y]$ and $G_x = G-N[x]$ are in $W_2$. Clearly, $H_i$ is a component of $G_y$, for $i \neq 2$, and $H_j$ is a component of $G_x$, for $j \neq 1$. Hence, $H_i$ is a $W_2$ graph for all i. It follows that G-v is also a $W_2$ graph. []

In order to consider triangle-free $W_2$ graphs, we introduce some terminology given in [9]. A 5-cycle in a graph is called a basic 5-cycle provided that it contains no two adjacent points of degree $\geq 3$ (that is, at most two points in the 5-cycle can have degree $\geq 3$ and two such points must be nonadjacent). A graph G is in the family PC if V(G) can be partitioned into two sets V(C) and V(P) such that V(C) contains points from basic 5-cycles in G and V(P) contains points from pendant lines in G; in addition, the lines induced by V(P) must be independent. Two graphs in PC are given in Figure 1.

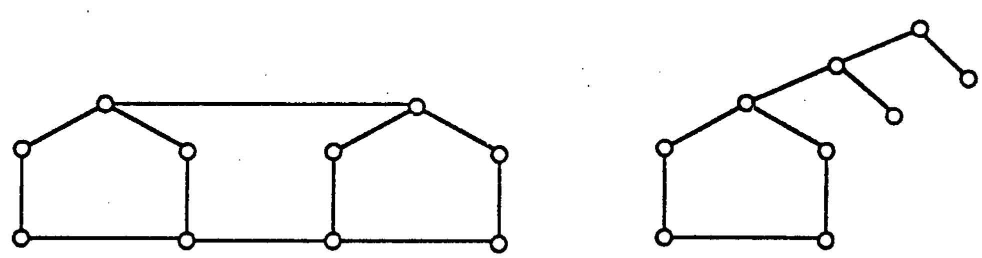

Figure 1

The girth of a graph is the size of a smallest cycle in the graph. We say a graph with no cycles has infinite girth. Finbow, Hartnell and Nowakowski [9] proved that the family PC described above contains all well-covered graphs with girth at least five, except $K_1$, $C_7$, and the four graphs shown in Figure 2. We state their result in the next theorem.

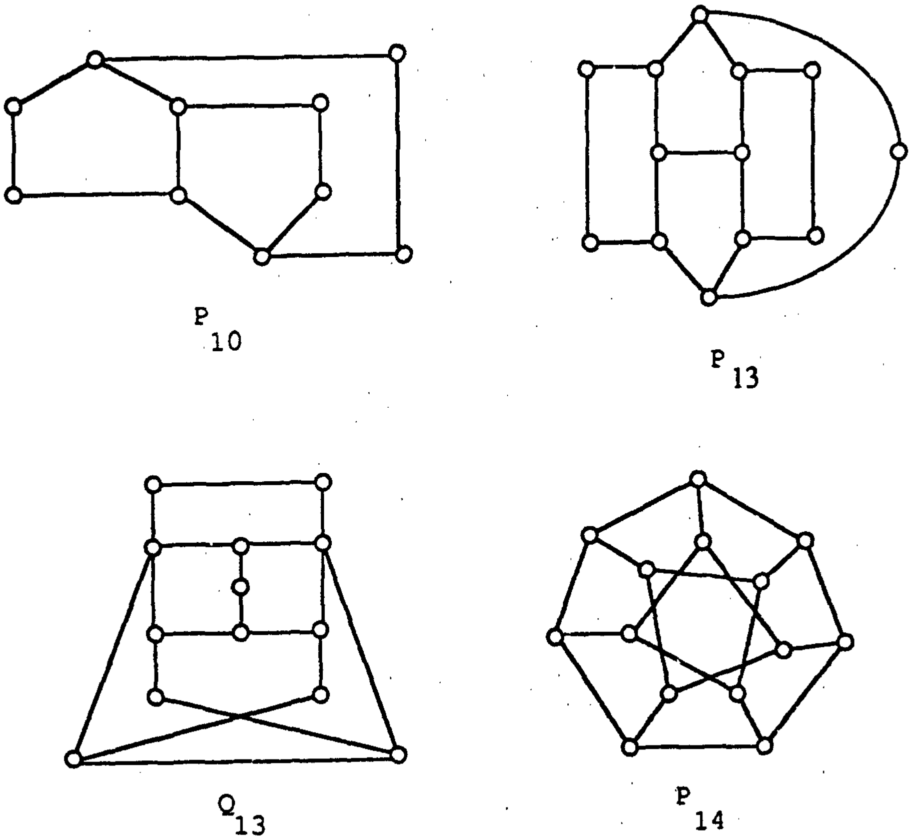


Figure 2

**Theorem 5.** Suppose G is well-covered with girth $\geq 5$. Then $G \in PC$ or $G \in \{K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$.

We need Lemma 6 to show that $K_2$ and $C_5$ are the only $W_2$ graphs in PC. Consequently, we prove in Theorem 7 that a $W_2$ graph other than $K_2$ and $C_5$ has girth at most four.

**Lemma 6.** If G is in PC with girth $\geq 5$ ($G \neq K_2$ or $C_5$), then $G \notin W_2$.

<u>Proof</u>. Suppose $G \in W_2$. By Theorem 1, we have $\delta \geq 2$. So $G \in PC$ and $\delta \geq 2$ together imply that $\{C_i\}$, $i = 1,\dots,n$, partitions $V(G)$, where each $C_i$ is a basic 5-cycle. Since $G \neq C_5$, then $i \geq 2$.

Now $C_1$ is joined to one or more of the $C_i$ ($i \geq 2$) by one or more lines. Without loss of generality, assume $C_1$ is connected to $C_2$ by line $e = uv$. Let $C_1 = uabcd$ and $C_2 = vwxyz$. Since $C_1$ is a basic 5-cycle then either v is not adjacent to b or v is not adjacent to c. We can assume that v is not adjacent to c. Since $v \sim u$, then $\deg(a) = 2$. Thus, $\{v,c\}$ is independent and so $\{v,c\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G, a contradiction since $G \in W_2$. []

**Theorem 7.** If $G \in W_2$ ($G \neq K_2$ or $C_5$), then girth $G \leq 4$.

Proof. Suppose girth $G \geq 5$ and G is well-covered. By the preceding lemma, if $G \in PC$ then $G \notin W_2$. From Theorem 5, if $G \notin PC$, then $G \in \{K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$. It is straightforward to check that each of these 6 graphs is not in $W_2$ by finding a pair of disjoint independent sets that do not extend to disjoint maximum independent sets. Thus, if G is well-covered with girth $\geq 5$, then $G \notin W_2$. []

Hence, a $W_2$ graph (other than $K_2$ and $C_5$) must contain a triangle or a 4-cycle. Thus, a triangle-free $W_2$ graph (other than $K_2$ and $C_5$) has girth 4. We study $W_2$ graphs of girth four for the remainder of this paper.

A line in a graph G is a critical line if its removal increases the independence number. A line-critical graph is a graph with only critical lines. Staples proved in [14] that a triangle-free $W_2$ graph is line-critical. Hence, all graphs given in the following constructions are line-critical.

## CONSTRUCTIONS

The following constructions show how to build a larger (in size and independence number) $W_2$ graph of girth four from a given such graph with some additional properties. The fact that the constructions yield $W_2$ graphs can be verified directly from the definition of a $W_2$ graph by showing that every two disjoint independent sets can be extended to two disjoint maximum independent sets.

**Construction 1.** Suppose H is a $W_2$ graph of girth 4 and C is a 4-cycle in H such that $\alpha(H\text{-}C) = \alpha(H) - 1$ and H-C is in $W_2$. Let C = acbd and let xy be a new line and A = $v_1v_2v_3v_4$ be a new 4-cycle. Form a new graph G with

$V(G) = V(H) \cup V(A) \cup \{x,y\}$, and

$E(G) = E(H) \cup E(A) \cup \{xy, v_1x, v_3y, v_2a, v_2b, v_4c, v_4d\}$. See Figure 3.

Then G is a $W_2$ graph of girth 4 with $\alpha(G) = \alpha(H) + 2$.

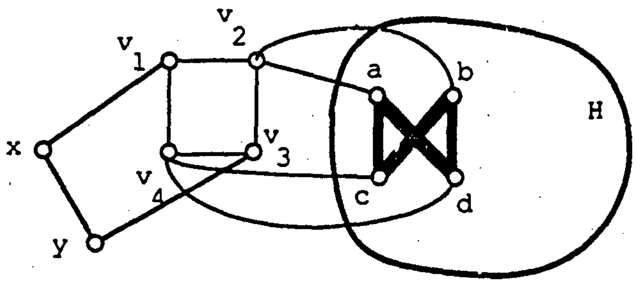


Figure 3

Suppose $H_1$ is the graph in Figure 4. If C is the 4-cycle in $H_1$, then $H_1$-C is a $W_2$ graph. Also, $\alpha(H_1\text{-}C) = 2 = \alpha(H_1) - 1$. Thus, we can use $H_1$ to construct a larger $W_2$ graph of girth 4 with independence number 5 via the construction in Construction 1. Call this graph $G_5$.

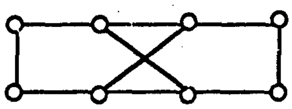

Figure 4

Let $H_2$ be the graph on 12 points in Figure 6. Let C be the 4-cycle acbd, as indicated in Figure 6. It can be checked that $H_2$ is a $W_2$ graph with $\alpha(H_2) = 4$, and $H_2$-C = $H_1$. Thus, $\alpha(H_2\text{-}C) = \alpha(H_2) - 1$, and $H_2$-C is a $W_2$ graph. Thus, we can build a larger $W_2$ graph of girth 4 (with independence number 6) from $H_2$ via the construction in Construction 1. Call this graph $G_6$.

Let $H_1 = G_3$. Note that $G_5$ satisfies the conditions in Construction 1, with the 4-cycle A that was used to build $G_5$ from $G_3$ satisfying $\alpha(G_5\text{-}A) = \alpha(G_5) - 1$ and $G_5\text{-}A \in W_2$. Hence, we can obtain a $W_2$ graph of girth 4 from $G_5$, call it $G_7$, via the construction in Construction 1. Therefore, by starting with $H_1 = G_3$ we can recursively use the construction in Construction 1 to generate an infinite family of $W_2$ graphs of girth 4, namely $G_3, G_5, G_7, G_9, \ldots$, where $\alpha(G_n) = n$, for all odd n. Note that the "new" 4-cycle used to construct $G_{2k+1}$ from $G_{2k-1}$ is a 4-cycle in $G_{2k+1}$ which satisfies the conditions in Construction 1. Thus, we "attach" to this 4-cycle to construct $G_{2k+3}$ from $G_{2k+1}$ via the construction in Construction 1. Similarly, by starting with $H_2 = G_4$, we can recursively generate $W_2$ graphs of girth 4, namely $G_4, G_6, G_8, G_{10}, \ldots$, where $\alpha(G_n) =$ n, for all even n.

By the nature of the construction in Construction 1, all graphs in the two infinite families just given are exactly 2-connected. In order to construct 3-connected and 4-connected $W_2$ graphs of girth 4, we develop a different construction in Construction 2.

**Construction 2.** Suppose H is a $W_2$ graph of girth 4 with disjoint 4-cycles $C_1$ and $C_2$ such that (i) $\alpha(H\text{-}C_i) = \alpha(H) - 1$, for i = 1, 2, and (ii) $H\text{-}C_i$ is a $W_2$ graph, for i = 1, 2. Also, H is either connected or has exactly two components. In the disconnected case, each component contains exactly one of the 4-cycles $C_i$.

Let $C_1 = u_1y_1v_1x_1$, $C_2 = u_2y_2v_2x_2$, and let A = abcd be a new 4-cycle. Form a new graph G with

$V(G) = V(H) \cup V(A)$, and

$E(G) = E(H) \cup E(A) \cup \{au_1, av_1, cx_1, cy_1, bx_2, by_2, du_2, dv_2\}$. See Figure 5.

Then G is a $W_2$ graph of girth 4 and $\alpha(G) = \alpha(H) + 1$.

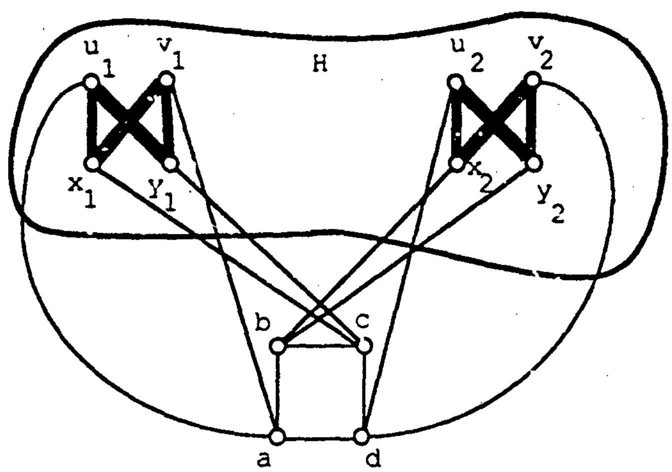


Figure 5

Note that in Construction 2, we allowed H to be the disjoint union of two $W_2$ graphs of girth 4, say $G_1$ and $G_2$, each containing a 4-cycle $C_i$ such that $G_i\text{-}C_i$ is a $W_2$ graph and $\alpha(G_i\text{-}C_i) = \alpha(G_i) - 1$, for i = 1, 2. In this case, $\alpha(G) = \alpha(G_1) + \alpha(G_2) + 1$ and G is exactly 2-connected.

Let H be the graph on 12 points given in Figure 6. It is straightforward to verify that H is a $W_2$ graph of girth 4. Let $C_1$ = uyvx and $C_2$ = acbd; then $C_1$ and $C_2$ are disjoint

4-cycles in H. $H$-$C_i$ is isomorphic to the graph in Figure 4, for $i = 1, 2$. Thus, $H$-$C_i$ is a $W_2$ graph and $\alpha(H\text{-}C_i) = \alpha(H) - 1$, for $i = 1, 2$.

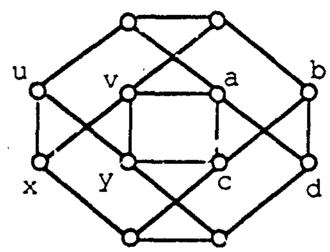


Figure 6

We will work with copies of H. For copy $H_i$, we will denote the 4-cycles corresponding to $C_1$ and $C_2$ by $C_{1,i} = u_iy_iv_ix_i$ and $C_{2,i} = a_ic_ib_id_i$, respectively.

Let $H_1$ and $H_2$ be two copies of H. Obtain a new graph $F_1$ by adjoining a new 4-cycle $A_1$ to $C_{1,1}$ and $C_{2,1}$ as in Construction 2. See Figure 7. By Construction 2, graph $F_1$ is a $W_2$ graph of girth 4 and $\alpha(F_1) = 2\alpha(H_i) + 1$.

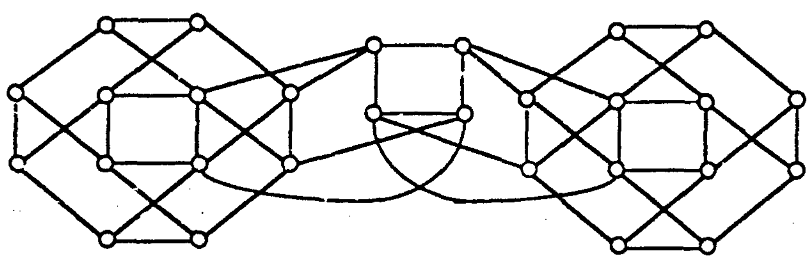

Figure 7

Since $H_1$-$C_{1,2}$ is a $W_2$ graph, then by Construction 2 the graph $F_1$-$C_{1,2}$ is also a $W_2$ graph. Moreover, $\alpha(F_1\text{-}C_{1,2}) = \alpha(H\text{-}C_{1,2}) + \alpha(H_2) + 1 = (\alpha(H_1) - 1) + \alpha(H_2) + 1 = \alpha(H_1) + \alpha(H_2) = \alpha(F_1) - 1$. Clearly, $F_1$-$A_1$ is a $W_2$ graph and $\alpha(F_1\text{-}A_1) = \alpha(F_1) - 1$. So we form a new graph $F_{1,1}$ from $F_1$ by adjoining a new 4-cycle $A_2$ to $C_{1,2}$ and $A_1$ by the construction in Construction 2. By Construction 2, graph $F_{1,1}$ is a $W_2$ graph of girth 4 with $\alpha(F_{1,1}) = \alpha(F_1) + 1$.

Clearly, $F_{1,1}\text{-}A_2$ is in $W_2$ and $\alpha(F_{1,1}\text{-}A_2) = \alpha(F_{1,1}) - 1$. Since $H_2\text{-}C_{2,2}$ is in $W_2$, then by Construction 2 the graph $F_{1,1}\text{-}C_{2,2}$ is in $W_2$. Also, $\alpha(F_{1,1}\text{-}C_{2,2}) = \alpha(F_1\text{-}C_{2,2}) + 1 = (\alpha(F_1) - 1) + 1 = \alpha(F_1) = \alpha(F_{1,1}) - 1$. So we form a new graph $F_{1,2}$ by adjoining a new 4-cycle $A_3$ to $A_2$ and $C_{2,2}$ by the construction given in Construction 2. Let $G_1 = F_{1,2}$. $G_1$ is shown in Figure 8. Then $G_1$ is a $W_2$ graph of girth 4 by Construction 2. Also, $G_1$ is 3-connected, $|V(G_1)| = 36$ and $\alpha(G_1) = 2\alpha(H_i) + 3 = 11$.

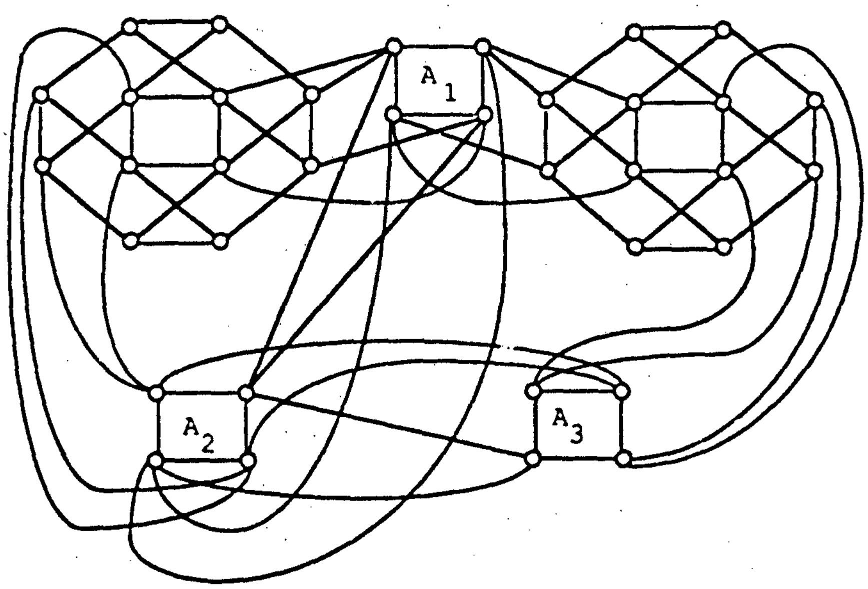


Figure 8

We conjecture that it is possible to construct an infinite family of 3-connected $W_2$ graphs of girth 4 by using Construction 2 and a technique generalized from that used to construct the graph $G_1$ given above.

Beginning with H given above in Figure 6, we can obtain the graph H' given in Figure 9 by two successive applications of Construction 2. Thus, H' is a $W_2$ graph of girth 4. Note that H' is 4-connected. We conjecture that it is possible to construct an infinite family of 4-connected $W_2$ graphs of girth 4 by using Construction 2 and by having H' play the role of H above in constructing $G_1$.

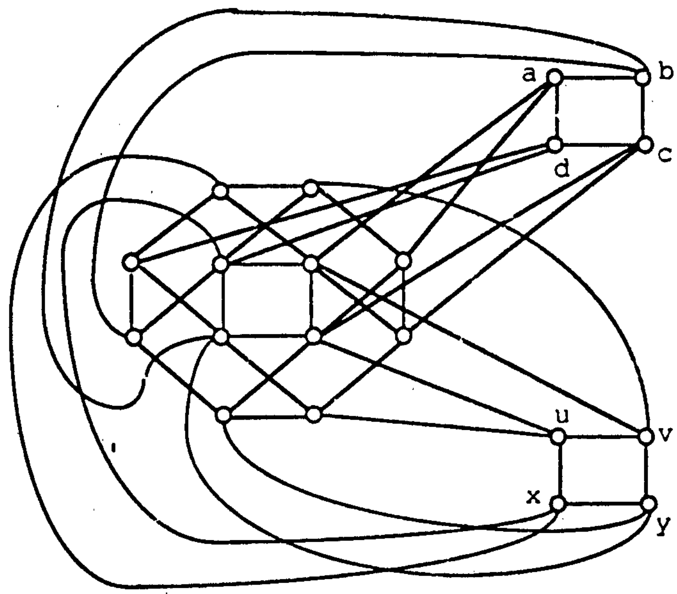


Figure 9

Not all $W_2$ graphs of girth 4 arise from the constructions given above. Neither of the graphs given in Figure 10 can be built using our constructions. The graph on 13 points is 4-regular and was found by Royle [12] using a computer program. Note that neither of the graphs has any 4-cycle that satisfies the conditions in Construction 1 or Construction 2.

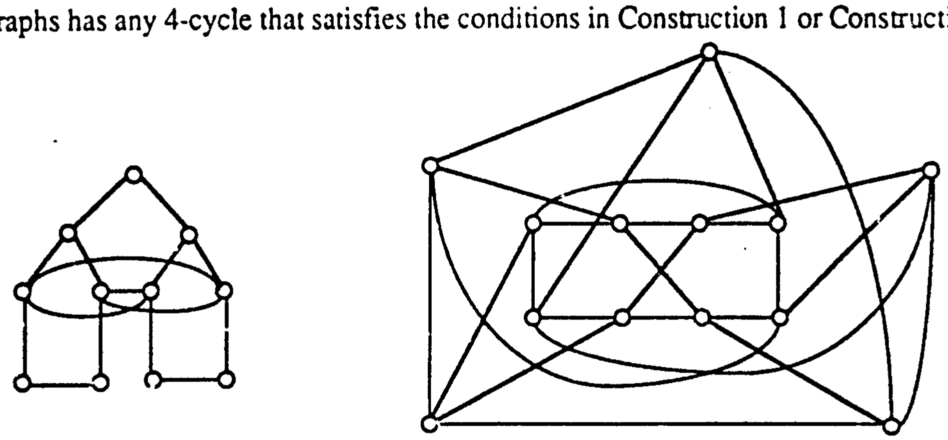

Figure 10

## CUTSETS

Now that we have constructed some $W_2$ graphs of girth 4, we look at minimum point cutsets for such graphs.

**Theorem 8**. If G is a $W_2$ graph of girth 4, then G is 2-connected.

Proof. Assume to the contrary that G has a cutpoint v. Let $G_1, G_2, \ldots, G_n$ be the components of G-v. By Theorem 4, graphs $G_1, \ldots, G_n$ are $W_2$ graphs. Let $N_i = N(v) \cap G_i$, for $i = 1,\ldots,n$. Since G has girth 4, then $N_i$ is independent for all i. Since $G_i \in W_2$, there exists maximum independent sets $J_i$ in $G_i$ such that $J_i \cap N_i = \varnothing$, for all i. Clearly, $J = J_1 \cup \ldots \cup J_n$ is an independent set in G. Consequently, J and $\{v\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G. This is a contradiction since $G \in W_2$. Hence, G is 2-connected. []

**Lemma 9**. Suppose G is a $W_2$ graph of girth 4 and $\{u,v\}$ is a cutset of G. If $u \sim v$, then every component of $G-\{u,v\}$, except possibly one, is a $W_2$ graph.

Proof. Let $G_1, \ldots, G_n$ be the components of $G-\{u,v\}$. Let $U_i = N(u) \cap G_i$ and $V_i = N(v) \cap G_i$, for all i. Since G has girth 4, then $x \in U_i$ implies x is not adjacent to v, and $y \in V_i$ implies y is not adjacent to u, for all i. Also, $U_i$ and $V_i$ are independent sets, for all i.

Suppose that $x \in U_i$, $y \in V_i$ implies $x \sim y$, for all i. Let $U = U_1 \cup \ldots \cup U_n$. Then U and $\{v\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus, there exists $j \in \{1, \ldots, n\}$ such that x and y are points in $G_j$, $x \in U_j$, $y \in V_j$ and x is not adjacent to y.

Consider the graph $G_x = G-N[x]$. Since x is not adjacent to y, then $y \in G_x$. Since $G \in W_2$, then by Theorem 3 so is $G_x$. Then v is a cutpoint for $G_x$, and by Theorem 4, the graph $G_x$-v is a $W_2$ graph. Since $G_i$ is a component of $G_x$-v, for $i \neq j$, then $G_i$ is a $W_2$ graph, $i \neq j$. []

Theorem 10. Suppose G is a $W_2$ graph of girth 4 and $\{u,v\}$ is a cutset for G. Then $\{u,v\}$ is independent.

Proof. Suppose $u \sim v$. Let $G_1, \ldots, G_n$ be the components of $G-\{u,v\}$. Let $U_i = N(u) \cap G_i$ and $V_i = N(v) \cap G_i$, for all i. Since G has girth 4, then $U_i$ and $V_i$ are disjoint independent sets, for all i.

Case 1. Suppose $G_i$ is a $W_2$ graph, for all i. Then there exist maximum independent sets $J_i$ in $G_i$ such that $J_i \supseteq V_i$ and $J_i \cap U_i = \varnothing$, for all i. Let $J = J_i \cup \ldots \cup J_n$. Then J and $\{u\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G, contradicting $G \in W_2$.

Case 2. So $G_j$ is not a $W_2$ graph, for some j. By Lemma 9, graph $G_i$ is a $W_2$ graph for $i \neq j$. So let $J_i \supseteq V_i$ be a maximum independent set in $G_i$ such that $J_i \cap U_i = \varnothing$, for all $i \neq j$. For each $i \neq j$, pick $x_i \in U_i$. Let $X = \{x_i\colon\ i \neq j\}$. Clearly X is an independent set. By Theorem 3, the graph $G_X = G-\{X \cup N(X)\}$ is a $W_2$ graph.

Suppose there exists some $y \in V_i$ such that y is not adjacent to $x_i$, for some $i \neq j$. Then v is a cutpoint for $G_X$. By Theorem 4, the graph $G_X$-v is in $W_2$. Since $G_X$-v contains $G_j$ as a component and $G_j$ is not a $W_2$ graph, we obtain a contradiction. Thus, $y \in V_i$ implies $y \sim x_i$, for all $i \neq j$.

Let H be the subgraph of G induced by $G_j \cup \{v\}$. Since $y \in V_i$ implies $y \sim x_i$, for all $i \neq j$, then H is a component of $G_X$. Since $G_X \in W_2$, then $H \in W_2$. Hence, there exists maximum independent set $J_j$ in H such that $J_j \supseteq V_j$ and $J_j \cap U_j = \varnothing$. Let $J = J_1 \cup \ldots \cup J_n$. Then J and $\{u\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G. This contradicts $G \in W_2$.

Therefore $\{u,v\}$ must be independent. []

Since a cutset of size two in a $W_2$ graph of girth 4 is independent, we are led to ask if the same is true for minimum cutsets of size three or more. The next two lemmas help to answer the question for minimum cutsets of size three in $W_2$ graphs of girth 4.

**Lemma 11.** Suppose G is 3-connected $W_2$ graph of girth 4 and $\{u,v,t\}$ is a cutset for G. Then $\{u,v,t\}$ does not induce exactly one line.

Proof. Assume to the contrary that $\{u,v,t\}$ induces precisely the line uv. Let $G_1, \ldots, G_n$ be the components of $G-\{u,v,t\}$. Let $U_i = N(u) \cap G_i$, $V_i = N(v) \cap G_i$, and $T_i = N(t) \cap G_i$, for all i.

Since t is adjacent to neither u nor v, then we must have $t \sim x$ for all $x \in U_i$ and $t \sim y$ for all $y \in V_i$, for all values of i except possibly one. Otherwise, the graph $G_t$ is a $W_2$ graph with cutset $\{u,v\}$, contradicting Theorem 10. Without loss of generality, we assume $t \sim x$ for all $x \in U_i$ and $t \sim y$ for all $y \in V_i$, for all $i \neq 1$. Since G has no triangles, it follows that the sets $U_i \cup V_i$ are independent, for $i \neq 1$.

Consider any component different from $G_1$, say $G_2$. Choose $s \sim t$ such that $s \in V_2$. Then the graph $G_s$ has u as a cutpoint. So by Theorem 4, graph $G_s$-u is a $W_2$ graph. Since $G_1$ is a component of $G_s$-u, then $G_1$ is a $W_2$ graph.

Case 1. Suppose there exists $a \in U_1$ and $b \in V_1$ such that $a \sim t$ and $b \sim t$. Since G has no triangles, then a is not adjacent to b. Thus, $G_a$ is a $W_2$ graph which has v as a cutpoint. By Theorem 4, the graph $G_a$-v is a $W_2$ graph. Since $G_i$, $i \neq 1$, is a component of $G_a$-v, then $G_i$, $i \neq 1$, is a $W_2$ graph. Thus, there exist maximum independent sets $J_i$ in $G_i$ such that $J_i \cap V_i = \varnothing$ $(i \neq 1)$, and there exists a maximum independent set $J_1$ in $G_1$ such that $a \in J_1$ and $J_1 \cap V_1 = \varnothing$. Let $J = J_1 \cup \ldots \cup J_n$. Then J and $\{v\}$ are independent sets in G which don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$.

Case 2. So either t is not adjacent to a for all $a \in U_1$, or t is not adjacent to b for all $b \in V_1$. Without loss of generality, assume t is not adjacent to a for all $a \in U_1$.

Case 2.1. Suppose $T_1$-$V_1 \neq \varnothing$. Let $x \in T_1$-$V_1$; that is, $t \sim x$ and $x \notin V_1$. From the assumption t is not adjacent to a for all $a \in U_1$, we see also that $x \notin U_1$. If there exists $a \in U_1$ such that x is not adjacent to a, or $b \in V_1$ such that x is not adjacent to b, then the $W_2$ graph $G_x$ has {u,v} as a cutset. This contradicts Theorem 10. Thus, $x \sim a$ for all $a \in U_1$ and $x \sim b$ for all $b \in V_1$. Since G has girth 4, then $a \in U_1$ and $b \in V_1$ imply that a is not adjacent to b. Similarly, $b \in V_1$ implies b is not adjacent to t. Hence, $T_1 \cap V_1 = \varnothing = T_1 \cap U_1$. Therefore, $T_1$-$V_1 = T_1$. Thus, if $y \in T_1$, it follows that $y \sim a$ for all $a \in U_1$.

Fix $z \in U_1$. From above, $z \sim y$ for all $y \in T_1$. But then $G_z$ has v as a cutpoint and so by Theorem 4 the graph $G_z$-v is a $W_2$ graph.

Case 2.1.1. Suppose $n \geq 3$. Then t is a cutpoint for $G_z$-v. By Theorem 4, graph $G_z$-v-t is a $W_2$ graph. Since $G_i$, $i \neq 1$, is a component of $G_z$-v-t, then $G_i$, $i \neq 1$, is a $W_2$ graph. Thus, there exist maximum independent sets $J_i$ in $G_i$ such that $V_i \cap J_i = \varnothing$ $(i \neq 1)$, and there exists maximum independent set $J_1$ in $G_1$ such that $z \in J_1$ and $V_1 \cap J_1 = \varnothing$. Let $J = J_1 \cup \ldots \cup J_n$. Then J and {v} don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$.

Case 2.1.2. So assume $n = 2$. Let H be the graph induced by $G_2 \cup t$. Then H is a component of $G_z$-v; hence, H is a $W_2$ graph. From earlier, $U_2$ and $V_2$ are disjoint and independent, and $t \sim x$ for all $x \in U_2$. Since H is a $W_2$ graph, there exists maximum independent set $J_H$ in H such that $J_H \supseteq U_2$ and $J_H \cap V_2 = \varnothing$. Note that $t \notin J_H$. Since $G_1$ is a $W_2$ graph, there exists maximum independent set $J_1$ in $G_1$ such that $z \in J_1$ and $J_1 \cap V_1 = \varnothing$. Then $J = J_1 \cup J_H$ is independent in G. So J and {v} don't extend to disjoint maximum independent sets in G, a contradiction.

Case 2.2. Thus, $T_1$-$V_1 = \varnothing$. Hence, $V_1 \supseteq T_1$. Since $U_1$ and $V_1$ are disjoint independent sets in $G_1$, then there exists maximum independent set $J_1 \supseteq U_1$ in $G_1$ such that $J_1 \cap V_1 = \varnothing$. But then $J_1 \cup \{t\}$ and {v} are disjoint independent sets in G which don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$.

Therefore, {u,v,t} does not induce exactly one line in G. []

**Lemma 12.** Suppose G is a 3-connected $W_2$ graph of girth 4 with cutset $\{u,v,t\}$. Then $\{u,v,t\}$ induces at most one line.

Proof. Assume to the contrary that $\{u,v,t\}$ induces two lines, say uv and vt (since G has girth 4, then $\{u,v,t\}$ cannot induce three lines). Let $G_1, \ldots, G_n$ be the components of $G-\{u,v,t\}$. Let $U_i = N(u) \cap G_i$, $V_i = N(v) \cap G_i$ and $T_i = N(t) \cap G_i$, for all i. Note that $U_i \cap V_i = \emptyset = V_i \cap T_i$, for all i; however, we do not know that $U_i \cap T_i = \emptyset$.

Suppose for all $x \in U_i$, for all i, that $t \sim x$. Then $\{t\}$ and $\{u\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. So, without loss of generality, assume there exists some $x \in U_1$ such that t is not adjacent to x.

Suppose there exists some $z \in V_1$ such that z is not adjacent to x. Then $G_x$ has $\{v,t\}$ as a cutset, contradicting Theorem 10. We are implicitly using Theorem 13 here, which states that a $W_2$ graph of girth 4 is 2-connected. Hence, $z \in V_1$ implies $x \sim z$.

Since G is 3-connected, $T_1 \neq \emptyset$. If there exists some $y \in T_1$ such that x is not adjacent to y, then $G_x$ will have $\{v,t\}$ as a cutset, again contradicting Theorem 10. So $y \in T_1$ implies $y \sim x$. Since G has girth 4 and $x \sim y$ for all $y \in T_1$, then $U_1 \cap T_1 = \emptyset$. Since $x \sim y$ for all $y \in T_1$ and $x \sim z$ for all $z \in V_1$, then $y \in T_1$ and $z \in V_1$ implies y is not adjacent to z. Thus, if $y \in T_1$, then $G_y$ has $\{u,v\}$ as a cutset. This contradicts Theorem 10.

Therefore, $\{u,v,t\}$ cannot induce two lines in G. Since G has girth 4, it follows that $\{u,v,t\}$ induces at most one line. []

With the two preceding lemmas, it is a simple matter to prove in the next theorem that a minimum cutset of size three in a $W_2$ graph of girth 4 must be independent.

**Theorem 13**. If G is a 3-connected $W_2$ graph of girth 4 with cutset $\{u,v,t\}$, then $\{u,v,t\}$ is independent.

Proof. By Lemma 12, the set $\{u,v,t\}$ induces at most one line. By Lemma 11, the set $\{u,v,t\}$ does not induce exactly one line. Hence, $\{u,v,t\}$ induces no lines in G; that is, $\{u,v,t\}$ is independent in G. []

For minimum cutsets of size four or greater, we do not know if the cutsets must be independent.

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